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An Adaptive Echo Canceller in a Nonideal Environment (Nonlinear or Time Variant)

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In this paper we calculate a lower bound on suppression provided by an adaptive echo canceller in either a nonlinear or time variant environment. Specifically, we examine the effects on performance of nonlinear echo paths described by a Volterra integral equation [*equation* (18)] *or time variant echo paths caused by phase jitter on single-sideband suppressed carrier systems.*

I. INTRODUCTION

In a previous paper[1] we have described the performance of an adaptive echo canceller in a linear time invariant environment. Qualitatively speaking, the environment in which an echo canceller will operate appears to be mainly linear and time invariant; however, some echo paths will be nonlinear and/or time variable. Some typical causes of nonlinearity are the volume dependent gain in compandored circuits† and harmonic distortion in amplifiers and repeaters. Time variability, on the other hand, may be caused by spurious modulation of the carrier of long-haul single-sideband suppressed carrier systems. This is commonly referred to as incidental FM or phase jitter. In certain cases these anomalies are of sufficient magnitude to degrade the performance of an adaptive echo canceller.

In this paper we examine the operation of an adaptive echo canceller studied previously in an ideal environment[1] in a nonideal environment (nonlinear, time variant). We restrict ourselves to nonlinearities which do not possess infinite memory and to time variability caused by phase jitter. In both cases we derive a lower bound on the suppression provided by the echo canceller and empirically verify

† This problem occurs when the compressor portion of the compandor is not perfectly compensated for by the expandor.

our results. For the sake of brevity we restrict our discussion to a digital implementation of the type shown in Fig. 1. However, one can investigate other implementations of the echo canceller and obtain similar results.

II. GENERAL CONSIDERATIONS AND NOTATIONAL FORMS

In Fig. 1 we show how an echo canceller would be connected in a typical connection. The input signal $x(t)$ produces an echo $y(t)$ corrupted by noise, $\zeta(t)$. An approximation of $y(t)$, $y_A(t)$, is subtracted from the actual echo and noise producing a cancelled echo $e(t)$. Examining the echo canceller in a little more detail we find that it is composed of M digital filters having the set of orthonormal impulse responses $\{\lambda_i(k)\}$, and the set of outputs $\{w_i(k)\}$ to input $x(k)$.† Every tap has associated with it the adaptive network shown for two taps in the figure. From Fig. 1 we conclude that the gain of the ith tap at the k + 1 sampling interval is given by

$$g_i(k + 1) = g_i(k) + |K|\, e(k)w_i(k). \tag{1}$$

Let us assume that the response of the echo path to $x(t)$ may be given in the following form:

$$y(t) = x(t) * h(t) + \psi(t). \tag{2}$$

where $h(t)$ is any square integrable function, (*) denotes convolution, and $\psi(t)$‡ is any function which is required to make equation (2) correct. Qualitatively speaking we see from equation (2) that we are breaking $y(t)$ into a linear component $x(t) * h(t)$ and a distortion term $\psi(t)$. In subsequent sections of this paper we will demonstrate that the nonlinear or time variant echo paths that we choose to study yield $y(t)$'s of the form of equation (2). We will therefore first examine the effect of a system described by equation (2) on the performance of an adaptive echo canceller. We will then calculate $\psi(t)$ for a nonlinear or a time variant echo path.

2.1 *The Governing Equation*

Since we hypothesized that $h(t)$ was a square integrable function we may represent it by a generalized Fourier series

$$h(t) = \sum_{i=1}^{\infty} C_i \lambda_i(t)$$

† $x(k)$ denotes the value of $x(t)$ at the kth sampling interval; also, all sampling is assumed to be at the Nyquist rate.

‡ $\psi(t)$ may depend on the absolute time when $x(t)$ is applied.

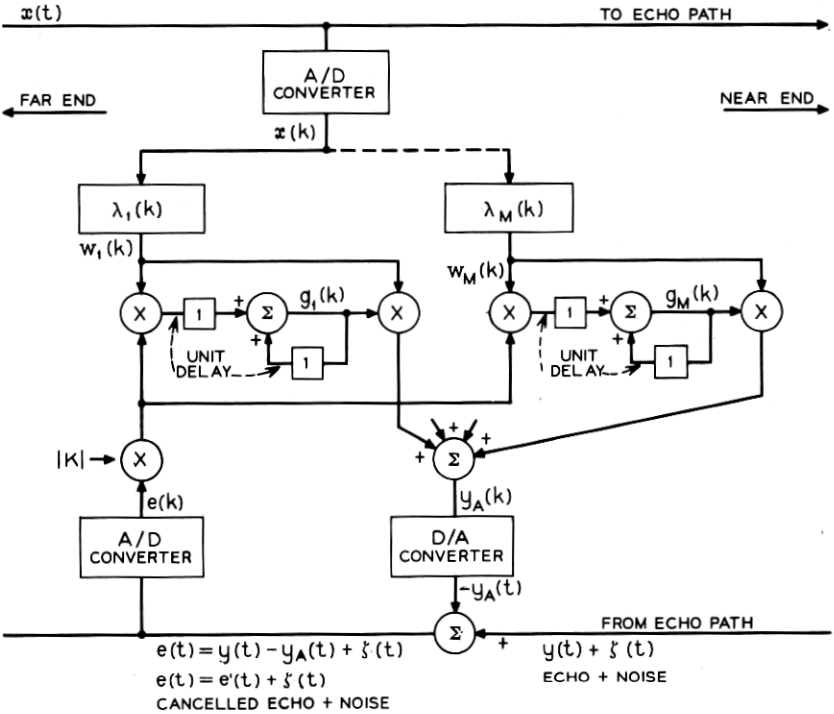

Fig. 1—An adaptive echo canceller.

where

$$C_i = \int_0^\infty h(t)\lambda_i(t)\,dt.$$

Substituting $\sum_{i=1}^\infty C_i\lambda_i(t)$ for $h(t)$ in equation (2) we obtain

$$y(t) = \sum_{i=1}^\infty C_i w_i(t) + \psi(t)$$

where $w_i(t) = x(t) * \lambda_i(t)$.

From Fig. 1 we see that

$$y_A(t) = \sum_{i=1}^M g_i(t) w_i(t).$$

Therefore,

$$e(t) = \sum_{i=1}^M r_i(t) w_i(t) + \sum_{i=M+1}^\infty C_i w_i(t) + \psi(t) + \zeta(t) \tag{3}$$

where

$$r_i(t) = C_i - g_i(t). \tag{4}$$

Combining equations (1) and (4) we obtain

$$r_i(k+1) = r_i(k) - | K | w_i(k)e(k).$$

Squaring the above equation, summing over the M taps of the echo canceller, and using equation (3) we obtain

$$\begin{aligned}\sum_{i=1}^{M} (r_i^2(k+1) - r_i^2(k)) = -2 | K | \sum_{i=1}^{M} r_i(k)w_i(k)\Bigg(\sum_{j=1}^{M} r_j(k)w_j(k) \\ + \sum_{j=M+1}^{\infty} C_j w_j(k) + \psi(k) + \zeta(k)\Bigg) \\ + | K |^2 \sum_{i=1}^{M} w_i^2(k)\Bigg(\sum_{j=1}^{M} r_j(k)w_j(k) \\ + \sum_{j=M+1}^{\infty} C_j w_j(k) + \psi(k) + \zeta(k)\Bigg)^2 \cdot \end{aligned} \tag{5}$$

In order to simplify our notation we make the following three definitions:

$$\theta(k) \equiv \sum_{i=M+1}^{\infty} C_i w_i(k) + \psi(k), \tag{6a}$$

$$u^2(k) \equiv \sum_{i=1}^{M} w_i^2(k), \tag{6b}$$

$$v(k) \equiv \sum_{i=1}^{M} r_i(k)w_i(k). \tag{6c}$$

As a result we obtain

$$\begin{aligned}\sum_{i=1}^{M} (r_i^2(k+1) - r_i^2(k)) \\ = -2 | K | (v^2(k) + \theta(k)v(k) + \zeta(k)v(k)) \\ + | K |^2 u^2(k)\{v^2(k) + \theta^2(k) + \zeta^2(k) + 2\theta(k)v(k) \\ + 2\zeta(k)v(k) + 2\zeta(k)\theta(k)\}.\end{aligned} \tag{7}$$

The above equation describes the performance of the adaptive echo canceller shown in Fig. 1 in the environment described by equation (2). However, before we can apply equation (7) to the problem at hand one final definition is required. We define the average value of a sampled function $f(k)$ in the interval $Qp + 1 \leqq k \leqq (p + 1)Q$ as

$$\overline{f(p)} = \frac{1}{Q} \sum_{k=Qp+1}^{(p+1)Q} f(k) \qquad p = 0, 1, 2, \cdots \tag{8}$$

where Q is the number of samples per interval and is constant. Therefore, $\overline{f(p)}$ may be interpreted as the average value of $f(k)$ averaged over the pth interval containing Q samples. Note that for all p the $p - 1$ and $p + 1$ intervals are adjacent to and do not overlap the pth interval.

2.2 *Choices of | K | Which Allow Best Match to the Echo Path Fourier Coefficients*

Let us now apply our averaging technique defined by equation (8) to equation (7). We will assume that the circuit noise $\zeta(t)$ is zero mean, with variance σ_ζ^2. Also, we will assume that the noise is statistically independent of the other variables in equation (7). This is justified by virtue of the fact that we are assuming that K is small.[1] Furthermore, we will assume that the number of samples in the pth interval, Q, is large enough so that $\overline{\zeta(p)}$ and $\bar{\zeta}^2(p)$ are good estimates of the true mean and variance of the circuit noise. As a result of the above we obtain

$$\begin{aligned} 1/Q \sum_{i=1}^{M} \{r_i^2[(Qp + 1) + Q] - r_i^2(Qp + 1)\} &= -2 \mid K \mid \overline{v^2(p)} - 2 \mid K \mid \overline{\theta(p)v(p)} \\ &\quad + 2 \mid K \mid^2 \overline{\theta(p)v(p)u^2(p)} \\ &\quad + \mid K \mid^2 \overline{u^2(p)v^2(p)} + \mid K \mid^2 \sigma_\zeta^2 \overline{u^2(p)} \\ &\quad + \mid K \mid^2 \overline{\theta^2(p)u^2(p)}. \end{aligned} \tag{9}$$

From (9) it is clear that the sum of the r_i^2 is reduced when the right-hand side of the equation is negative.† In fact, it is continually reduced until the right-hand side of (9) can no longer be negative. Therefore a sufficient condition for convergence‡ can be written as

$$\begin{aligned} \Delta(p) = \mid K \mid^2 \overline{[(v(p) + \theta(p))^2 u^2(p) + \sigma_n^2 u^2(p)]} \\ - 2 \mid K \mid (\overline{v^2(p)} + \overline{\theta(p)v(p)}) < 0 \end{aligned} \tag{10a}$$

† This may be more easily seen if $\Sigma_{i=1}^{M} r_i^2$ is considered to be the magnitude squared of a vector whose components are $\{r_i\}$. In this event the left-hand side of the equation is proportional to the difference of the magnitude square of a vector at some instant and Q samples earlier. When the right-hand side is negative it indicates that the magnitude of the vector is smaller than it was Q samples ago.

‡ In this paper convergence is taken to mean the reduction of the sum of the r_i^2.

or equivalently

$$\Delta'(p) = |K| [\overline{(v(p) + \theta(p)^2 u^2(p) + \sigma_n^2 u^2(p))}] - 2\overline{(v^2(p) + \theta(p) v(p))} < 0. \tag{10b}$$

By defining

$$B(p) \equiv \overline{(V(p) + \theta(p))^2 U^2(p) + \sigma_n^2 U^2(p)},$$
$$D(p) \equiv \overline{(V^2(p) + \theta(p) V(p))},$$

we find

$$\Delta(p) = |K|^2 B(p) - 2|K| D(p).$$

From the above equation it should be clear that

$$\Delta(p) \leqq 0$$

for

$$0 < |K| < 2 \left| \frac{D(p)}{B(p)} \right|$$

and

$$D(p) \geqq 0.$$

Therefore we conclude that the echo canceller converges for $D(p) > 0$ and $0 \leqq |K| \leqq 2D(p)/B(p)$. However, since $|V(p)|$ is small in the neighborhood of equilibrium, $D(p)$ decreases quicker than $B(p)$ and as a result the upper bound on $|K|$ is reduced while convergence is taking place. Convergence ceases when

$$|K| = \frac{2D(p)}{B(p)}.$$

On the other hand for $D(p) < 0$ there is no positive choice of K which will allow convergence and therefore the canceller diverges until $D(p)$ becomes positive and $0 < |K| \leqq |2D(p)/B(p)|$. As a result the best choice of $|K|$ which will allow the most convergence (the smallest $|D(p)|$ and therefore the best match to the Fourier coefficients C_i) is the smallest $|K|$. Ideally $|K| \to 0$ will allow the best match to the echo path Fourier coefficients. However, the smaller $|K|$ is made the longer it takes for the canceller to converge, and in practice a compromise is required between amount and speed of convergence.

2.3 *A Lower Bound on Achievable Suppression*

We have shown previously (Section 2.2) that choosing $|K|$ very small will allow the best match of the tap gains $g_i(k)$ to the Fourier coefficients C_i. Taking the limit of equation (10b) as $|K|$ goes to zero we obtain

$$\lim_{|K|\to 0} \Delta'(p) = -\overline{2(v^2(p)} + \overline{\theta(p)v(p))} \leqq 0$$

or convergence takes place as long as

$$\overline{v^2(p)} > -\overline{\theta(p)v(p)} \tag{11a}$$

and equilibrium is reached when

$$\overline{v^2(p)} = -\overline{\theta(p)v(p)}. \tag{11b}$$

A sufficient condition for (11) to be true is

$$\overline{v^2(p)} \geqq |\ \overline{\theta(p)v(p)}\ |.$$

From the definition of $\overline{[\cdot]}$ [equation (8)] and applying the Schwarz inequality, we obtain

$$|\ \overline{\theta(p)v(p)}\ | \leqq \overline{|\ \theta(p)v(p)\ |} \leqq (\overline{v^2(p)})^{\frac{1}{2}}(\overline{\theta^2(p)})^{\frac{1}{2}}.$$

From the above two inequalities we see that a looser but still sufficient condition for (11a) and (11b) to be valid is

$$\overline{v^2(p)} \geqq (\overline{v^2(p)})^{\frac{1}{2}}[\overline{\theta^2(p)}]^{\frac{1}{2}}$$

or

$$\overline{v^2(p)} \geqq \overline{\theta^2(p)}. \tag{12}$$

Equation (12) is a sufficient condition to insure convergence in the pth interval. We therefore conclude that when $|K|$ is made infinitesimally small the echo canceller will reduce $\overline{\sum_{i=1}^{M} r_i^2(p)}$ at least until:

$$\overline{v^2(p)} \leqq \overline{\theta^2(p)}. \tag{13}$$

We will now use equations (13) and (11a) and (11b) to determine a lower bound on the achievable suppression. Referring to Fig. 1 we define the suppression S as

$$S \equiv 10 \log_{10} \frac{\overline{y^2(p)}}{\overline{(e'(p))^2}}. \tag{14}$$

From equations (3) and (6) it may be easily shown that

$$e'(k) = v(k) + \theta(k)$$

or

$$\overline{(e'(p))^2} = \overline{v^2(p)} + \overline{2v(p)\theta(p)} + \overline{\theta^2(p)}.$$

Substituting equation (11b) into the above equation we obtain

$$\overline{(e'(p))^2} = \overline{v(p)\theta(p)} + \overline{\theta^2(p)}.$$

Also, since

$$\overline{(e'(p))^2} \leqq |\overline{v(p)\theta(p)}| + \overline{\theta^2(p)}, \tag{15}$$

we can apply Schwarz's inequality to the above and obtain

$$\overline{(e'(p))^2} \leqq (\overline{v^2(p)})^{\frac{1}{2}}(\overline{\theta^2(p)})^{\frac{1}{2}} + \overline{\theta^2(p)}.$$

Substituting (13) and the above into (14) we obtain

$$S \geqq 10 \log_{10} \frac{\overline{y^2(p)}}{2\overline{\theta^2(p)}} = S_1 . \tag{16}$$

On the other hand, if for a given input signal

$$\overline{v(p)\theta(p)} = 0,^{\dagger}$$

we see from (15) and (14) that

$$S \geqq 10 \log_{10} \frac{\overline{y^2(p)}}{\overline{\theta^2(p)}} = S_2 . \tag{17}$$

Equations (16) and (17) are lower bounds on the achievable suppression which occurs when $|K|$ is chosen to be infinitesimal. For any other than infinitesimal $|K|$ the suppression may or may not be less than the values given by the lower bound [(16) and (17)]. However, as will be seen in a later section of this paper, for the choices of $|K|$ which allow reasonable settling times, (16) and (17) do form lower bounds on suppression.

III. NONLINEAR ECHO PATHS

We will now become more specific and consider the effect of a nonlinear echo path. We will restrict our discussion to a class of nonlinear echo paths which possess the following four properties:‡

(*i*) They are time invariant.
(*ii*) They are deterministic.
(*iii*) They are "smooth." Qualitatively speaking by smooth we mean that the echo path cannot introduce any abrupt or switch-like changes in the output. If such a change is evident in the output then it must be due to a similar switch-like change in the input.
(*iv*) They possess noninfinite memory. That is the memory does not depend on the remote past.

† For example, this occurs when θ and v are independent and one is zero mean. As will be seen this is the case for systems exhibiting phase jitter.

‡ Most echo paths encountered in practice satisfy all the conditions except one and these are handled separately in Section IV.

Echo paths which possess the above four properties may be conveniently characterized by a Volterra integral equation.[2–6] As a result, $y(t)$ shown in Fig. 1 is related to $x(t)$ by the equation given below:

$$y(t) = \int_0^\infty h_1(\tau_1)x(t-\tau_1)\,d\tau_1 + \int_0^\infty \int_0^\infty h_2(\tau_1, \tau_2)x(t-\tau_1)x(t-\tau_2)\,d\tau_1\,d\tau_2 + \sum_{n=0}^{\infty} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} h_n(\tau_1 \cdots \tau_n) \prod_{j=1}^{n} x(t-\tau_j)\,d\tau_j \,. \tag{18}$$

We may easily place equation (18) in the form of equation (2):

$$y(t) = x(t) * h_1(t) + \sum_{n=2}^{\infty} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} h_n(\tau_1 \cdots \tau_n) \prod_{j=1}^{n} x(t-\tau_j)\,d\tau_j \,.$$

Comparing the above equation with equation (2) we see that

$$\psi(t) = \sum_{n=2}^{\infty} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} h_n(\tau_1 \cdots \tau_n) \prod_{j=1}^{n} x(t-\tau_j)\,d\tau_j \,.$$

As a result we conclude that the bounds given by equations (16) and (17) are applicable to any nonlinear system possessing the properties outlined in the beginning of this section. We need only replace $\theta(k)$ by

$$\theta(k) = \sum_{i=M+1}^{\infty} C_i w_i(k) + \sum_{n=2}^{\infty} \underbrace{\int \cdots \int}_{n} h_n(\tau_1 \cdots \tau_n) \prod_{j=1}^{n} x(k-\tau_j)\,d\tau_j \,. \tag{19}$$

In this case, $\theta(k)$ is composed of that part of the linear portion of the echo path which the echo canceller cannot compensate for due to its limited number of taps and the nonlinear portion.

IV. TIME VARIABILITY

In this section we are specifically interested in time variability of the type predominantly found on single-sideband suppressed carrier systems and commonly referred to as incidental FM or phase jitter.†

† We use incidental FM and phase jitter interchangeably.

We will show that the effect of incidental FM on suppression can be calculated directly from the index of modulation, β, which is easily measured.

4.1 *Carrier System Model–Assumptions*

In Fig. 2 we show a block diagram of a typical single-sideband suppressed carrier system whose output will be corrupted by phase

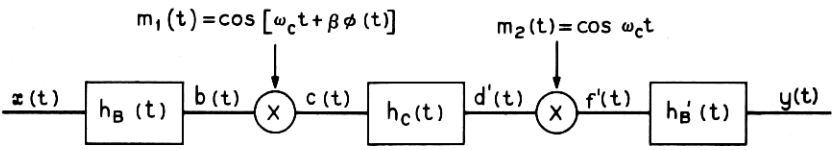

Fig 2—Single-sideband suppressed carrier system with phase jitter.

jitter. We will begin by making the following practical assumptions:

(*i*) The input signal $x(t)$ is bandlimited to ω_s. That is,

$$X(\omega) = \begin{cases} X(\omega) & \text{for} \quad |\omega| \leqq \omega_s{}^\dagger \\ 0 & \text{otherwise.} \end{cases}$$

(*ii*) The filter $h_B(t)$ is a low-pass filter bandlimited to ω_B,

$$H_B(\omega) = \begin{cases} H_B(\omega) & \text{for} \quad |\omega| < \omega_B \quad \text{where} \quad \omega_B > \omega_s + \omega_i \\ 0 & \text{otherwise.} \end{cases}$$

(*iii*) The incidental FM is narrow-band FM,

$$|\beta\phi(t)| \ll \pi/2 \ \forall t$$

and

$$\phi(t) = \cos \omega_i t \qquad \qquad \omega_i \ll \omega_s .$$

(*iv*) The filter $h_c(t)$ is an ideal bandpass filter, at carrier frequencies, which passes the upper sideband,

$$H_c(\omega) = \begin{cases} 1 & \text{for} \quad \omega_c \leqq |\omega| \leqq \omega_c + \omega_s + \omega_i \ , \omega_c \gg \omega_B \\ 0 & \text{otherwise.} \end{cases}$$

† Capital letters are reserved for functions of frequency, small letters for functions of time.

$\therefore$ $x(t)$ has the Fourier Transform $X(\omega)$.

(v) The filter $h_{B'}(t)$ is a low-pass filter bandlimited to ω_B,

$$H_{B'}(\omega) = \begin{cases} H_{B'}(\omega) & \text{for} \quad |\omega| < \omega_B \quad \text{where} \quad \omega_B > \omega_s + \omega_j \\ 0 & \text{otherwise.} \end{cases}$$

4.2 *Derivation of the Relationship Between $x(t)$ and $y(t)$*

Consider Fig. 3 for a moment. $H_+(\omega)$ is an ideal, not physically realizable, filter which passes only positive frequencies. That is,

$$H_+(\omega) = \begin{cases} 1 & \omega \geqq 0 \\ 0 & \omega < 0. \end{cases} \tag{20}$$

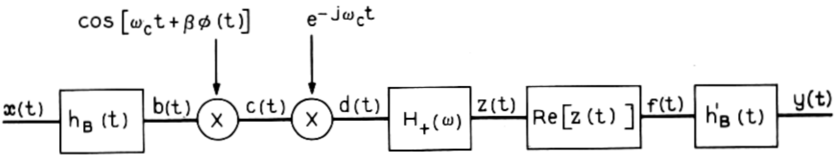

Fig. 3—Mathematically equivalent to Fig. 2.

and Re(·) signifies the real part of (·). It may be verified by the reader that Fig. 3 is mathematically equivalent to Fig. 2 as far as $x(t)$ and $y(t)$ are concerned. Therefore, we will use it for our analysis since it reduces the manipulations required. From Fig. 3,

$$c(t) = \frac{b(t)}{2}\left[e^{j(\omega_c t+\beta\phi(t))} + e^{-(j\omega_c t+\beta\phi(t))}\right] \tag{21}$$

and

$$d(t) = \frac{b(t)}{2}e^{j\beta\phi(t)} + \frac{b(t)}{2}e^{-j\beta\phi(t)}e^{-j2\omega_c t}. \tag{22}$$

Since the second term in equation (22) is a high-frequency term and its effect will be filtered out by the filter $h_{B'}(t)$ we will disregard it. Using the assumption $|\beta\phi(t)| \ll \pi/2$ we expand $e^{jB\phi(t)}$ obtaining

$$d(t) = \frac{b(t)}{2}(1 + j\beta\phi(t)). \tag{23}$$

Continuing, we obtain for $z(t)$†

$$z(t) = \tfrac{1}{2}[d(t) + j\,\widehat{d(t)}] = \frac{b(t)}{4} - \frac{\widehat{\beta\phi(t)b(t)}}{4} + j\widehat{\left(\frac{b(t)}{4} + \frac{\beta\phi(t)b(t)}{4}\right)}.^{\ddagger} \tag{24}$$

† The reader is referred to Ref. 7 for the proof of equation (24).

‡ $\widehat{[\cdot]}$ denotes Hilbert transform of $[\cdot]$.

Therefore,

$$\theta(t) = 1/4(b(t) - \beta\widehat{\phi(t)b(t)}) \tag{25}$$

and

$$y(t) = 1/4(b(t)*h_{B'}(t) - \beta(\widehat{\phi(t)b(t)})*h_{B'}(t)). \tag{26}$$

Comparing equations (2) and (26) we conclude that they are of the same form:

$$h(t) = 1/4h_B(t)*h_{B'}(t)$$

and

$$\psi(t) = 1/4\beta(\widehat{\phi(t)b(t)})*h_{B'}(t).$$

As a result the bounds given by equations (16) and (17) are applicable with $\theta(t)$ given by

$$\theta(t) = \sum_{i=m+1}^{\infty} C_i w_i(t) - 1/4\beta(\widehat{\phi(t)b(t)}) * h_{B'}(t). \tag{27}$$

However, unlike the nonlinear problem, we may simplify the above results considerably as shown below.

4.3 *Simplification of Bound for Systems Displaying Incidental FM*

If we assume that the basis set $\{\lambda_i(k)\}$ is complete we obtain from equation (27) that

$$\theta(t) = -1/4\ \beta(\widehat{\phi(t)b(t)})*h_{B'}(t).$$

For the situation where incidental FM is present we have found experimentally that quantity $\overline{v(p)\theta(p)}$ [see equation (6)] may be safely assumed to be zero. Therefore the bound S_2 given by equation (17) is applicable.

For bandlimited signals sampled at the Nyquist rate,

$$\frac{\overline{y^2(p)}}{\overline{\theta^2(p)}} \approx \frac{\lim\limits_{\tau_1\to\infty} 1/\tau_1 \int_0^{\tau_1} y^2(t)\,dt}{\lim\limits_{\tau_2\to\infty} 1/\tau_2 \int_0^{\tau_2} \theta^2(t)\,dt}.$$

Since the effect of incidental FM on the circuits that we are dealing with is a second-order effect and is quite small, it is reasonable to assume that

$$\overline{|\, b(t) * h_{B'}(t)\,|} \gg \overline{|\,\beta(\widehat{\phi(t)b(t)}) * h_{B'}(t)\,|}.$$

Therefore,

$$\lim_{\tau_1\to\infty} 1/\tau_1 \int_0^{\tau_1} y^2(t)\, dt \approx \frac{1}{16}\left(\lim_{\tau_1\to\infty} 1/\tau_1 \int_0^{\tau_1} (b(t) * h_{B'}(t))^2\, dt\right)$$

resulting in

$$\frac{\overline{y^2(p)}}{\overline{\theta^2(p)}} \approx \frac{1}{\beta^2}\frac{\lim_{\tau_1\to\infty} 1/\tau_1 \int_0^{\tau_1} [b(t) * h_{B'}(t)]^2}{\lim_{\tau_2\to\infty} 1/\tau_2 \int_0^{\tau_1} [(\widehat{\phi(t)b(t)}) * h_{B'}(t)]^2\, dt} = \frac{1}{\beta^2}\frac{\int_{-\omega_s-\omega_j}^{\omega_s+\omega_j} S_y(\omega)\, d\omega}{\int_{-\omega_s-\omega_j}^{\omega_s+\omega_j} S_\theta(\omega)\, d\omega} \tag{28}$$

where $S_y(\omega)$ and $S_\theta(\omega)$ are proportional to the power spectral density of $y(t)$ and $\theta(t)$ respectively.

We will now show that the two integral expressions in (28) are approximately equal. In the frequency domain we have

$$\begin{aligned} S_y(\omega) &= |\, B(\omega)\, |^2\, |\, H_{B'}(\omega)\, |^2, \\ S_\theta(\omega) &= |\, \mathfrak{F}[\widehat{\phi(t)b(t)}] H_{B'}(\omega)\, |^2 \,{}^\dagger \\ &= |\, \mathfrak{F}[\widehat{\phi(t)b(t)}]\, |^2\, |\, H_{B'}(\omega)\, |^2. \end{aligned} \tag{29}$$

The Hilbert transform does not affect the magnitude of the Fourier transform. As a result we obtain

$$S_\theta(\omega) = |\, \mathfrak{F}[\phi(t)b(t)]\, |^2\, |\, H_{B'}(\omega)\, |^2. \tag{30}$$

We will now apply assumption *iii* and assume that the phase jitter function $\phi(t)$ is a cosine wave,‡

$$\phi(t) = \cos \omega_j t$$

where ω_j is typically either 60 Hz, 80 Hz, or 120 Hz. Equation (30) then becomes

$$S_\theta(\omega) = 1/4\, |\, B(\omega - \omega_j) + B(\omega + \omega_j)\, |^2\, |\, H_{B'}(\omega)\, |^2. \tag{31}$$

Since the effect of $\phi(t)$ is only to shift the spectrum $B(\omega)$ to the right and left a very small amount§ relative to its bandwidth,§ it should be clear from equations (29) and (31) that

$$\int_{-\omega_s-\omega_j}^{\omega_s+\omega_j} S_y(\omega)\, d\omega \approx \int_{-\omega_s-\omega_j}^{\omega_s+\omega_j} S_\theta(\omega)\, d\omega. \tag{32}$$

† $\mathfrak{F}[\phi(t)b(t)]$ indicates the Fourier transform of $\phi(t)b(t)$.

‡ In practice this is a good assumption.

§ Recall $\omega_s \gg \omega_j$. Typically ω_s is 3500 Hz and $\omega_j \leqq 120$ Hz.

Using equations (28) and (32) in (17) we obtain

$$S \geqq 20 \log_{10} \frac{1}{\beta}.^{\dagger} \tag{33}$$

Equation (33) is what we have been striving for; it relates achievable suppression to the modulation coefficient of the unwanted incidental FM for an echo canceller with a complete basis set.

V. EMPIRICAL RESULTS

5.1 *Time Variant Systems (Incidental FM)*

Since it is difficult to accurately create a desired amount of incidental FM on a working L carrier system, an analog simulation was used. For all practical purposes it is identical to a real system. The only difference is that it allows a controlled amount of incidental FM to be inserted.

We chose flatly weighted noise bandlimited to 4 kHz as our input signal. A digital tape was made of the input and output of the L terminal simulator and this tape was used as input to a computer simulation of the adaptive echo canceller shown in Fig. 1. The results are shown in Fig. 4.

From Fig. 4 we see that there is good agreement between the empirical results and the analytical results for peak-to-peak phase jitter angles above 4 degrees. Below 4 degrees the echo canceller performance is limited by the fact that it utilizes a finite number of taps. Recall in our derivation of equation (33) we assumed that the echo canceller employed a complete basis set (i.e., infinite number of taps). This assumption begins to fail in the vicinity of 4 degrees peak-to-peak phase jitter and the performance of the echo canceller becomes limited by the incompleteness rather than by the incidental FM.

5.2 *Nonlinear Systems*

In order to verify our results we simulated the nonlinear echo path shown in Fig. 5. We used various combinations of input signals and feedback gain constants $|K|$ and in all cases the results were comparable to those shown in Table I for a white noise input.

A word of explanation is needed to clarify Table I. Suppression was calculated according to equation (14). The sampling rate was 0.1 ms and the averaging interval, Q, consisted of 501 samples, i.e., a 50.1-ms interval.‡ Forty such averages were computed during the two-

† β must be in radians.
‡ See equation (8).

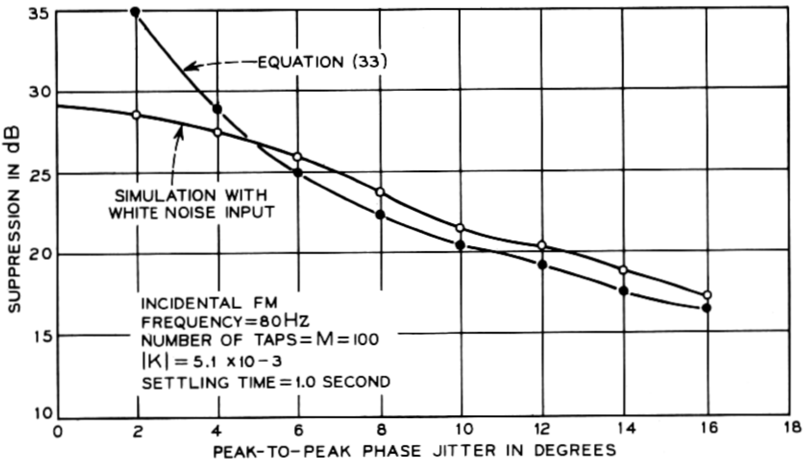

Fig. 4—Time variant simulation.

second time period after the settling time had elasped. The minimum and maximum suppression shown are those of the forty calculated suppressions, and the average suppression shown is the ensemble average of the forty calculated suppressions.

For all variation of input signals and feedback gain constants tested, we found that S_2 was 3 dB closer to the actual suppression than S_1 and within 5 dB of the actual suppression. However, as expected, the echo canceller always performed better than either bound.

VI. CONCLUDING REMARKS

We have obtained lower bounds on the suppression provided by an echo canceller in either a nonlinear or time invariant environment. Although theoretically the bounds are only valid when an infinitesimal feedback constant is used, we have found empirically that they apply for any practical choice of feedback gain constant. If one were able

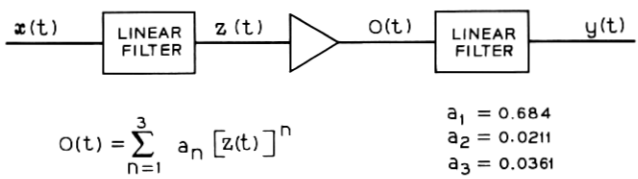

Fig. 5—Nonlinear simulation.

TABLE I—WHITE NOISE INPUT SIGNAL

Feedback gain $\|K\|$	5.1×10^{-3}
Settling time	1.0 second
Actual achieved suppression in dB	Min. = 18.2, max. = 22.1, ave. = 20.4
Bound S_1 [equation (16)]	Min. = 10.6, max. = 15.1, ave. = 13.7
Bound S_2 [equation (17)]	Min. = 13.6, max. = 18.1, ave. = 16.7

to measure the time variability of echo paths or to measure echo paths and then to separate the response into the linear and nonlinear portions [equation (18)], one could predict the minimum achievable suppression. Measurement of nonlinear echo paths has been the subject of study of others.[8,9]

For a nonlinear environment we have found that the nonlinear portion of the response and hence the lower bounds are dependent on the input signal. As a result, if this method is going to be used to predict the performance of an echo canceller for a speech input signal, it will be necessary to design an interrogation signal which possesses some of the same properties as does speech (syllabic rate, power level, peak-to-rms ratio, etc.)

The derived equations which relate the effect of incidental FM on suppression for the single-sideband suppressed carrier systems are sufficiently general to provide lower suppression bounds for different or more complex systems which exhibit incidental FM. One must merely know an effective index of modulation, β, measured between the input and the output. The calculations for more complex systems are rather involved and have been omitted.

Laboratory simulated echo cancellation for a nonlinear echo path and for an echo path with incidental FM produced cancellations somewhat better than the calculated lower bounds. Knowledge of the type of nonlinearity or time variability in the echo path environment may permit one to empirically increase the lower bounds derived in this paper.

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